VI. Finite/Infinite Dichotomy

In a cluster algebra, the fundamental data is cluster variables & quivers.
Natural question: How many are there?

<u>Defn</u>: $\mathcal{A}$ is of <u>finite type</u> if it has finitely many cluster variables (equivalently, finitely many seeds).
Otherwise, $\mathcal{A}$ is of <u>infinite type</u>.

<u>Example</u>: • Type A cluster algebras are of finite type

• The Markov cluster algebra
$$\mathcal{A}\left(\underline{x}, \begin{tikzcd}[sep=small] & \circ \arrow[dr, "2"] & \\ \circ \arrow[ur, "2"] & & \bullet \arrow[ll, "2"] \end{tikzcd}\right)$$
is infinite type

<u>Defn</u>: $\mathcal{A}$ is of <u>finite mutation type</u> if its seeds have finitely many distinct quivers.
Otherwise, $\mathcal{A}$ is of <u>infinite mutation type</u>.

<u>Obs:</u> Infinite mutation type

$\Downarrow$

Infinite type

(since there must be ∞ many seeds)

<u>Example:</u> • The Markov cluster algebra is finite mutation type

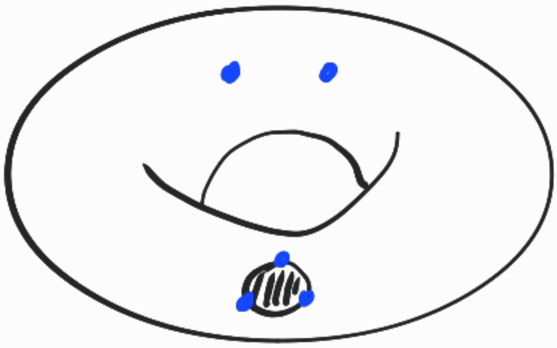
• Do alternating mutations at vertices 2 & 3 in middle example from warm-up

<u>Question:</u>

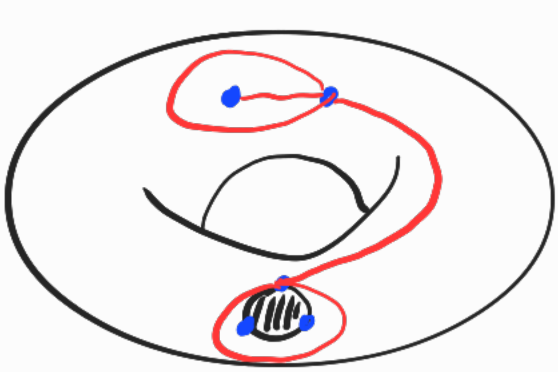
How can we tell the difference?

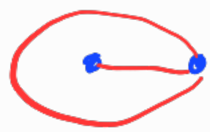
VII. Cluster Algebras from surfaces

Let S be a connected oriented surface with boundary, and M a finite set of marked points (excluding a few degenerate cases).



An <u>arc</u> γ in (S,M) is a non-self intersecting curve in S (up to isotopy) such that $\gamma \cap M =$ endpoints of γ, and γ does not cut out an unpunctured monogon or digon

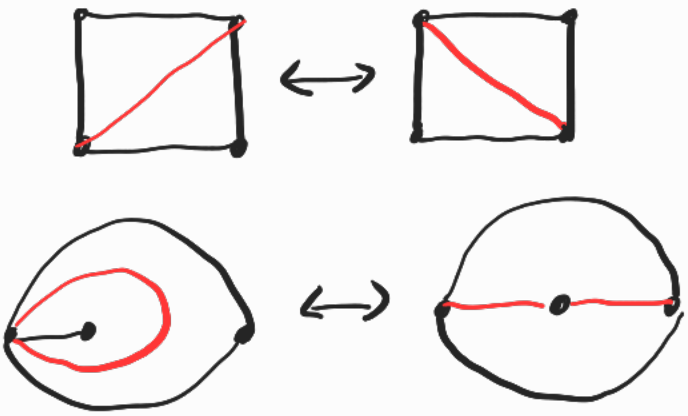




self-folded triangle

Arcs are <u>compatible</u> if they have nonintersecting realizations. Maximal collections of compatible arcs are <u>triangulations</u> of (S,M).

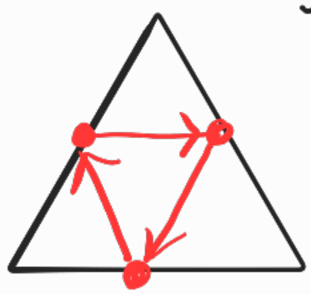
We can *flip* any non-boundary arc in (S,M) provided it is not inside a *self-folded triangle*.



flips

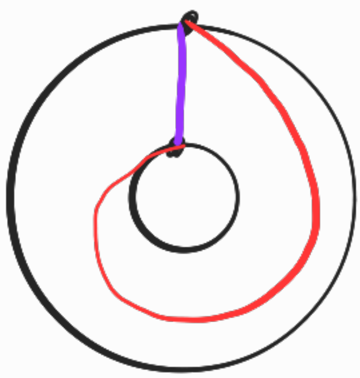
not flippable

If our triangulation T has no self-folded triangles, we can again get a quiver $Q(T)$ by inscribing a clockwise 3-cycle in each triangle with frozens at the boundary.

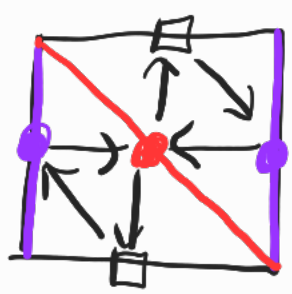


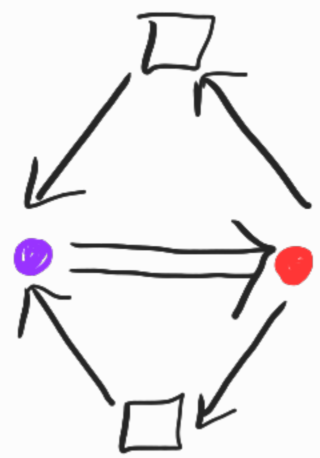
annulus

Ex:

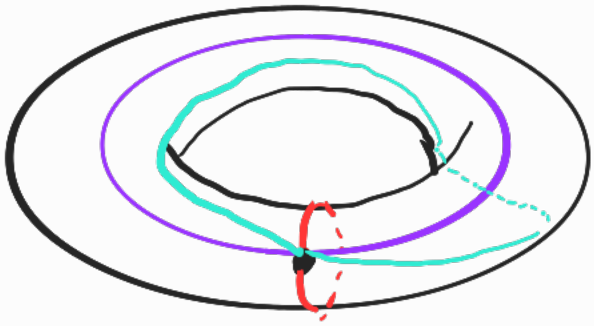


→ slice at purple arc

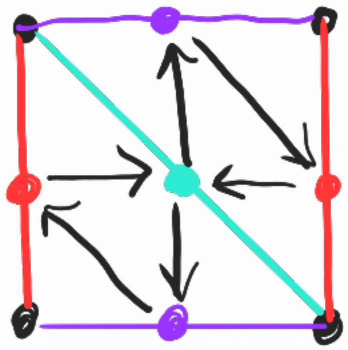


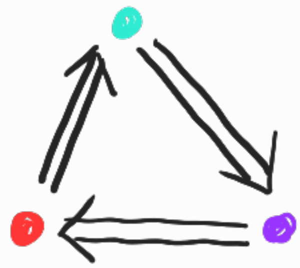


Ex:



→ slice at red & purple arcs





Markov!

Thm [Felikson-Shapiro-Tumarkin '08]

For a connected quiver Q of rank ≥ 3 with no frozens, the following are equivalent

- Q has finite mutation type
- Q comes from a connected triangulated surface or is one of 11 exceptional quivers

VIII. The Rest

- algebraic Lie theory (finite type classification)
- algebraic geometry (cluster varieties)
- quantum groups (canonical bases, quantum CAs)
- representations of quivers & finite-dimensional algebras

$$\left(\begin{tikzcd} & \mathbb{C}^2 \\ \mathbb{C}^3 \arrow[ur, "\alpha"] \arrow[r, "\beta"'] & \mathbb{C} \end{tikzcd}\right)$$

- Poisson geometry (integrable systems)
- Teichmüller theory (cluster variables are λ-lengths of arcs)
- Mirror symmetry (scattering diagrams, positivity)
- Scattering amplitudes (amplituhedron)
- . . . and more!